New integrable systems and a curious realisation of $SO(N)$

Jens Hoppe*

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Department of Mathematics
Royal Institute of Technology, Lindstedtsvägen 25 S-10044 Stockholm, Sweden
and
Center for Theoretical Physics
Seoul National University, 151-742 Seoul, Korea

Abstract

A multiparameter class of integrable systems is introduced.

*Email address: hoppe@math.kth.se

As is well known, geodesic motions on a manifold are integrable only in very special cases. Jacobi's ingenious solution for ellipsoids[1] and the impact of his invention of elliptical coordinates, explained in detail in his Königsberg lectures [2], can hardly be underestimated (reading through his five page note to the Prussian Academy [3], one feels that, despite so much progress during the century/ies that followed, (t)his kind of art is gone forever).

Comparatively recently ([4], see also [5] - [7]) the commutativity of the N quantities

$$F_i \equiv p_i^2 + \sum_{j=1}^{N}{}' \frac{(x_i p_j - x_j p_i)^2}{\alpha_i - \alpha_j} \tag{1}$$

for the ellipsoid ($\sum_{i=1}^{N} \frac{x_i^2}{\alpha_i} = 1$), resp.

$$G_i \equiv x_i^2 + \sum_{j=1}^{N}{}' \frac{(x_i p_j - x_j p_i)^2}{\alpha_i - \alpha_j} \tag{2}$$

for the related Neumann-problem [8] of N particles moving on the sphere ($\sum_{i=1}^{N} x_i^2 = 1$) subject to the external potential $V(\vec{x}) = \frac{1}{2}\sum \alpha_i x_i^2$, was noticed.

The commutativity with respect to

$$\{f, g\}(\vec{x}, \vec{p}) \equiv \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} \right) \tag{3}$$

follows straightforwardly from

$$\frac{1}{\alpha_{ik}\alpha_{kl}} + \frac{1}{\alpha_{kl}\alpha_{li}} + \frac{1}{\alpha_{li}\alpha_{ik}} = 0 \tag{4}$$

($\alpha_{ij} \equiv \alpha_i - \alpha_j$), and the standard angular momentum algebra relations,

$$[J_{ij}, J_{kl}] = -\delta_{jk} J_{il} + \delta_{ik} J_{jl} + \delta_{jl} J_{ik} - \delta_{il} J_{jk} \tag{5}$$

$$\begin{aligned} [J_{ij}, x_k] &= \delta_{ik} x_j - \delta_{jk} x_i \\ [J_{ij}, p_k] &= \delta_{ik} p_j - \delta_{jk} p_i \end{aligned} \tag{6}$$

satisfied by $J_{ij} \equiv x_i p_j - x_j p_i$ (and $[\ ,\ ] \equiv \{\ ,\ \}$).

In particular,

$$\begin{aligned} &\frac{1}{4}\left[\sum_j{}' \frac{L_{ij}^2}{\alpha_{ij}}, \sum_l{}' \frac{L_{kl}^2}{\alpha_{kl}} \right] \\ &= \sum_{j,l}{}'' \frac{L_{ij} L_{kl}}{\alpha_{ij}\alpha_{kl}} (-\delta_{jk} L_{il} + \delta_{ik} L_{jl} + \delta_{jl} L_{ik} - \delta_{il} L_{jk}) \\ &= \delta_{ik} \left(\sum_{j,l}{}'' \frac{L_{ij} L_{il} L_{jl}}{\alpha_{ij}\alpha_{il}} \right) - L_{ik} \sum_l{}'' L_{il} L_{kl} \left(\frac{1}{\alpha_{il}\alpha_{lk}} + \frac{1}{\alpha_{lk}\alpha_{ki}} + \frac{1}{\alpha_{ki}\alpha_{il}} \right) \end{aligned} \tag{7}$$

[1] announced 168 years ago in a letter to the French Academy [1].

is equal to zero as long as the L_{ij} satisfy (5).

The commutativity with respect to the constrained (Dirac) bracket,

$$\{f,g\}_{\mathrm{D}} \equiv \{f,g\} + \{f,\varphi\}\frac{1}{J}\{\Pi,g\} - \{f,\Pi\}\frac{1}{J}\{\varphi,g\}\ , \tag{8}$$

with $\varphi = \frac{1}{2}(\vec{x}^2 - 1)$, $\Pi = \vec{x}\cdot\vec{p}$, $(J \equiv \{\varphi,\Pi\} = 1)$ for the Neumann-problem, and $\varphi = \frac{1}{2}(\sum_{i=1}^{N}\frac{x_i^2}{\alpha_i} - 1)$, $\Pi = \sum_{i=1}^{N}\frac{x_i p_i}{\alpha_i}$, $(J \equiv \{\varphi,\Pi\})$ for the N-dimensional ellipsoid, then follows by noting the respective commutativity of φ with the F_i, resp. G_i.

Let me point out that (5) and (6) also imply the commutativity of the quantities

$$J_i = \alpha x_i p_i + {\sum_j}' \frac{(x_i p_j - x_j p_i)^2}{\alpha_i - \alpha_j}\ , \tag{9}$$

as for $i \neq k$ (and anticipating the quantum-commutativity by keeping track of the order)

$$\left[x_k p_k, {\sum_j}' \frac{J_{ij}^2}{\alpha_{ij}}\right] = x_k \frac{J_{ik}}{\alpha_{ik}} p_i + x_i \frac{J_{ik}}{\alpha_{ik}} p_k + x_k p_i \frac{J_{ik}}{\alpha_{ik}} + \frac{J_{ik}}{\alpha_{ik}} x_i p_k \tag{10}$$

is symmetric under the interchange of i and k, as well as note the formal commutativity of the differential operators $(k = 1, \cdots, N)$

$$\hat{H}_k = -{\sum_l}' (x_k x_l)^{1/4}(\partial_k - \partial_l)\frac{\sqrt{x_k x_l}}{\alpha_{kl}}(\partial_k - \partial_l)(x_k x_l)^{1/4} - \frac{i\alpha}{2}(x_k\partial_k + \partial_k x_k) \tag{11}$$

(acting on functions of $\vec{x} \in \mathbf{R}_+^N$, $\alpha \in \mathbf{R}$ and arbitrary $\alpha_{kl} = -\alpha_{lk} \neq 0$, satisfying (4)). The operators (with their classical counterparts $L_{ij} \equiv -2\sqrt{x_i x_j}(p_i - p_j)$)

$$\hat{L}_{ij} \equiv 2(x_i x_j)^{1/4}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)(x_i x_j)^{1/4} \equiv 2\rho_{ij}\partial_{ij}\rho_{ij} \tag{12}$$

satisfy the $so(N)$ Lie-algebra relations (5) (from now on, [,] denoting the ordinary commutator), as well as

$$[x_k\partial_k, \hat{L}_{ij}] = (\delta_{kj} - \delta_{ki})\left((\partial_i + \partial_j)\rho_{ij}^2 + \rho_{ij}^2(\partial_i + \partial_j)\right)\ , \tag{13}$$

$$\begin{array}{c} [\hat{L}_{ij}, x_k] = 2\rho_{ij}^2(\delta_{ik} - \delta_{jk}) \\ [\partial_k, \hat{L}_{ij}] = \frac{1}{4}(\delta_{ki} + \delta_{kj})\left(\frac{1}{x_k}\hat{L}_{ij} + \hat{L}_{ij}\frac{1}{x_k}\right) \end{array} \tag{14}$$

$$[\hat{L}_{ij}, \rho_{kl}^2] = \delta_{ik}\rho_{jl}^2 \pm 3 \text{ more}\ . \tag{15}$$

The commuting classical quantities corresponding to (11) are $(k = 1, \cdots, N)$

$$\tilde{H}_k = \sum_{l \neq k} \frac{x_k x_l}{\alpha_{kl}}(p_k - p_l)^2 + \alpha x_k p_k\ , \tag{16}$$

resp. (interchanging $\vec{x}$ and $\vec{p}$).

$$H_k = \sum_{l \neq k} p_k \frac{(x_k - x_l)^2}{\alpha_{kl}} p_l - \alpha x_k p_k \ . \tag{17}$$

Arbitrary functions of the commuting quantities (17) (resp. (16)/(11) or (9)) can be taken as Hamiltonians.

Let me finish with some remarks:

- While ($x_{ij} \equiv x_i - x_j$)

$$\left[x_k \partial_k + \partial_k x_k, {\sum_j}' x_{ij} \frac{\partial_i \partial_j}{\alpha_{ij}} x_{ij} \right] - (i \leftrightarrow k) = 0 \ , \tag{18}$$

  the naive quantisations of $(17)_{\alpha=0}$ do *not* commute:

$$\begin{aligned} &\left[{\sum_j}' x_{ij} \frac{\partial_{ij}^2}{\alpha_{ij}} x_{ij}, {\sum_l}' x_{kl} \frac{\partial_{kl}^2}{\alpha_{kl}} x_{kl} \right] \\ &= \frac{x_{ik}}{\alpha_{il}\alpha_{kl}} (\partial_i + \partial_k - \partial_l) + \frac{x_{kl}}{\alpha_{ki}\alpha_{li}} (\partial_k + \partial_l - \partial_i) + \frac{x_{li}}{\alpha_{lk}\alpha_{ik}} (\partial_l + \partial_i - \partial_k) \ . \end{aligned} \tag{19}$$

  Any other (hermitian) ordering of the 4 quantities x_{ij}, ∂_i, ∂_j, ∂_{ij} gives the same result (This discrepancy compared to the commutativity of (11), is due to having singled out the coordinate-representation, resp. avoiding $\sqrt{\partial_i \partial_j}$.).

- For $N = 2$, ($\alpha = 0$) and the choice $H = \sum_k \frac{H_k}{\alpha_k} = \frac{1}{2} \sum_{k,l} p_k \frac{x_{kl}^2}{\alpha_k \alpha_l} p_l$, e.g., one would get (with $q \equiv x_2 - x_1$, $p \equiv \frac{p_2 - p_1}{2}$, $P \equiv p_1 + p_2 = \text{const.}$, $\mu \equiv \alpha_1 \alpha_2$),

$$H = \frac{1}{2\mu} \left(\frac{P^2}{4} q^2 - p^2 q^2 \right) \ , \tag{20}$$

$$\ddot{q} = \frac{\dot{q}^2}{q} + \frac{P^2}{4\mu^2} q^3 \ , \tag{21}$$

  resp.

$$\dot{q}^2 = \frac{P^2}{4\mu^2} q^4 - \frac{2E}{\mu} q^2 \ . \tag{22}$$

  The integration is elementary.

- Apart from questions of domains, (11) is equivalent to the quantisation of (9).

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